§1. Motivation:

Recall : Moduli pb is a stack

$$\mathcal{X} : (Sch_S)^{op} \longrightarrow Groupoids$$

Goal: find M s.t.: $\mathcal{X}(T) = Hom(T, M)$

E.g.:

① $\mathcal{X}(T) = \left\{ \begin{array}{l} \text{groupoid of flat families} \\ \text{on } X \text{ over } T \\ + \text{ ample line bdl} \end{array} \right\}$

② $\mathcal{X}(T) = \left\{ \text{V.B. on } X \times T \right\}$

③ given a t-structure $D^b(coh(X))$

$$\mathcal{X}(T) = \left\{ \begin{array}{l} \text{flat families of objects} \\ \text{in } D^b(coh(X))^{\heartsuit} \text{ over } T \end{array} \right\}$$

④ $\mathcal{X}(T) = \left\{ \begin{array}{l} G\text{-bdls} : P \longrightarrow T \ \& \ G\text{-equivariant} \\ \text{map } T \to X \end{array} \right\}$ $\Big|$ G-reductive, $G \curvearrowright X$

④ is called the "quotient stack"

denoted $[X/G] \rightsquigarrow X/G$

e.g. $\boxed{\mathbb{A}^1/\mathbb{G}_m =: \Theta}$

- $\mathfrak{X}$ can be non sep. or non q-compact.
- $\mathfrak{X}$ can fail to be of finite type

<u>Goal</u>: a) Throw away some points and keep $\mathfrak{X}^{ss} \subset \mathfrak{X}$

b) $\mathfrak{X}^{ss}$ have a good moduli space

$\boxed{\text{c) Stratification of } \mathfrak{X}^{us}}$

<u>E.g.</u> C, smooth proj curve

$G = GL_n$, $Bun(C)$

- $Bun(C)$ is not q-compact.
- $Bun_{r,d}(C)$ ———
- $\forall r > 1$, no point of $Bun(C)_{r,d}$ is closed

and no connected component is q-compact

But b) is true

<u>Th</u>: • $\mathrm{Bun}^{ss}(C) \subset \mathrm{Bun}(C)$ is algebraic, q-compact and admits projective good mod. space

$$\mathrm{Bun}_{r,d}(C)^{ss} \longrightarrow M_{r,d}(C)$$

• Every ss E admits a Jordan-Holder filtration whose associated graded object is polystable.

<u>Th [Harder-Narasimhan]</u>: Let E be unstable V.B. then E contains $0 = \subsetneq E_p \subsetneq \dots \subsetneq E_0 = E$ s.th

$$\begin{cases} gr_i(E) \text{ loc. free, ss, } \forall i \\ \nu(gr_i(E)) \text{ is strictly increasing with } i. \end{cases}$$

<u>Proof sketch</u>: prove:

- E: has a max $F \subset E$ of maximal slope.
- E/F is loc free
- $E_1 := F$

$\boxtimes$

$\rightsquigarrow$ doesn't generalise

$\rightsquigarrow$ need $\neq$ approach

$\rightsquigarrow$ notion of filtration in arbitrary stacks.

$$\mathcal{X} = \mathcal{X}^{ss} \sqcup S_0 \sqcup \dots \sqcup S_n .$$

§1 – Notation:

- $\hookrightarrow$ (resp $\hookrightarrow$) are open (resp. closed immersion)
- G : reductive group
- B : base stack, loc. Noetherian.
- $\mathcal{X}_T := \mathcal{X} \times_B T$, T : B-stack.
- $\mathcal{X}$ verifies (+) : $\mathcal{X}$ is alg, l.f.t. over B, q-affine diagonal.
- $Y \in \mathrm{Top}$, $\mathrm{Irred}(Y)$: set of irred. components of Y.

§ 2. Framework : C : smooth proj. curve over k.

– Rees construction :

$$\left\{ \begin{array}{c} \text{Diagrams of sheaves on } C \\ \cdots \longrightarrow E_{w+1} \xrightarrow{\cdot t} E_w \longrightarrow \cdots \\ \text{each map is injective} \end{array} \right\}^{\star} \longleftrightarrow \left\{ \begin{array}{c} \text{Graded } \mathcal{O}_C[t]\text{-mod} \\ \text{flat over } k[t] \\ \text{with } t \text{ having} \\ \text{weight } -1 \end{array} \right\}$$

$$\cdots \longrightarrow E_{w+1} \xrightarrow{\cdot t} E_w \longrightarrow \cdots \longmapsto \bigoplus_{w \in \mathbb{Z}} E_w$$

$$\left\{ \begin{array}{c} \text{q-coh sheaves} \\ \text{on} \\ \mathbb{A}^1 \times C / \mathbb{G}_m \\ \wr\| \\ \Theta \times C \end{array} \right\} = \left\{ \begin{array}{c} \mathbb{G}_m\text{-equivariant} \\ \text{q-coh sheaves} \\ \text{on } \mathbb{A}^1 \times C \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Graded q-coh} \\ \mathcal{O}_C[t]\text{-modules} \end{array} \right\}$$

a diagram like $\star$ is "$\mathbb{Z}$-weighted filtrations"

$\rightsquigarrow$ $\mathbb{Z}$-weighted filtered loc. free sheaf is equivalent to a map $\Theta \longrightarrow \mathrm{Bun}(C)$

<u>Def:</u> $\mathcal{X}$ stack over S. a scheme.

$$\mathrm{Filt}(\mathcal{X}) : T \longmapsto \underline{\mathrm{Map}}_S(\Theta_T, \mathcal{X}), \quad \Theta_T = \Theta \times_S T$$

$$\mathrm{Grad}(\mathcal{X}) : T \longmapsto \underline{\mathrm{Map}}_S((B\mathbb{G}_m)_T, \mathcal{X})), \quad B\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}/\mathbb{G}_m$$

Θ has 2 points:
- generic point $\longleftrightarrow 1 \in \mathbb{A}^1$
- special point $\longleftrightarrow 0 \in \mathbb{A}^1$.

<u>Def:</u> Restrict $f: \Theta_k \longrightarrow \mathrm{Bun}(C)$ to $\{1\} \hookrightarrow \Theta_k$

$\rightsquigarrow$ taking $(\cdots E_{w+1} \longrightarrow E_w \longrightarrow \cdots) \mapsto \bigcup_w E_w$

In general $f : \Theta_k \longrightarrow \mathcal{X}$ seen as a filtration of $f(1) \in \mathcal{X}(k)$

$$\rightsquigarrow \quad ev_1 : \mathrm{Filt}(\mathcal{X}) \longrightarrow \mathcal{X}$$
$$f \longmapsto f(1)$$

- Restrict $f : \Theta_k \longrightarrow \mathcal{X}$ to $\{0/\mathbb{G}_m\} \hookrightarrow \Theta_k$

$\rightsquigarrow$ taking $(\cdots E_{w+1} \longrightarrow E_w \longrightarrow \cdots) \mapsto \bigoplus_w E_w/E_{w+1}$

In general, $f|_{\{0/\mathbb{G}_m\}} : B\mathbb{G}_m \to \mathcal{X}$ seen as

the graded object associated to f.

$\rightsquigarrow ev_0 : \mathrm{Filt}(\mathcal{X}) \longrightarrow \mathrm{Grad}(\mathcal{X})$.

- <u>Flag space</u>: Given a map $T \xrightarrow{\xi} \mathcal{X}$ over B

$$\mathrm{Flag}(\xi) := \underline{\mathrm{Map}}(\Theta, \mathcal{X}) \underset{ev_1, \mathcal{X}, T}{\times} T$$

<u>Prop.</u> $\mathrm{Filt}(\mathcal{X})$: Let $\mathcal{X}, \mathcal{Y}$ verifying $(+)$

let $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ repsentable by alg. spaces.

then $\mathrm{Filt}(\phi) : \mathrm{Filt}(\mathcal{X}) \longrightarrow \mathrm{Filt}(\mathcal{Y})$ is

the same.

- ϕ mono $\Longrightarrow$ $\mathrm{Filt}(\phi)$ is mono
- ϕ closed immersion $\Longrightarrow$ so is $\mathrm{Filt}(\phi)$
- ϕ open —— $\Longrightarrow$ ——————
- ϕ smooth (resp etale) $\Longrightarrow$ —————— and $\mathrm{Grad}(\phi)$

Def: • Θ-stratum: an open + closed substack

$S \subset \mathrm{Filt}(\mathcal{X})$ s.t. $\mathrm{ev}_1 \; S \hookrightarrow \mathcal{X}$.

• Θ-stratification: of $\mathcal{X}$ indexed by totally ordered set Γ

① $(\mathcal{X}_{\leq c} \hookrightarrow \mathcal{X})_{c \in \Gamma}$ s.t.

$\mathcal{X}_{\leq c} \hookrightarrow \mathcal{X}_{\leq c'}$ for $c < c'$ and $\mathcal{X} = \bigcup_{c \in \Gamma} \mathcal{X}_{\leq c}$

② a Θ-stratum $S_c \hookrightarrow \mathrm{Filt}(\mathcal{X})$ s.t.

$$\mathcal{X}_{\leq c} \setminus \mathrm{ev}_1(S_c) = \bigcup_{c' < c} \mathcal{X}_{\leq c'}$$

③ $\forall x \in |\mathcal{X}|$, the set $\{c \in \Gamma \mid x \in |\mathcal{X}_{\leq c}|\}$ has a minimal element.

Rem: • if Γ is well ordered then ③ holds automatically.

• $(S_c)_{c \in \Gamma}$ classify the unstable V.B. along with HN filtration.

• $\forall f \in S_c$, $f(0) = \mathrm{gr}(f) \in \mathcal{X}_{\leq c}$

- "weak Θ-stratif" $\rightsquigarrow$ replacing "$\hookrightarrow$" by "finite + radicial $\hookrightarrow$"

E.g.: HN-filtration of moduli of coh sheaves on a proj. scheme is a Θ-stratif.

[Nitsure, 2009] "on the schematic HN stratif".

Def: Sps $\{-\infty\} := \min \Gamma$.

$\mathcal{X}^{ss} := \mathcal{X}_{\leq \{-\infty\}}$ the ss locus.

$\mathcal{X}^{us} := |\mathcal{X}| \setminus |\mathcal{X}^{ss}|$

Lemma-def: Let $\mathcal{X}$ verify $(\dagger)$, let $\{S_{\leq c}\}_{c \in \Gamma}$ be a weak stratif of $\mathcal{X}$.

Then $\forall p \in \mathcal{X}(k)^{us}$, $\exists! c \in \Gamma$, $\exists! f \in |S_c|$

s.t. $p \in |\mathcal{X}_{\leq c}|$ and $f(1) = p$.

<u>Proof:</u> $|S_c| \xhookrightarrow[ev_1]{+} |\mathcal{X}_{\leq c}| \xhookrightarrow{\partial} |\mathcal{X}|$

is a loc closed immersion.

$\rightsquigarrow |S_c| \subset |\mathcal{X}|$

(2) $\Rightarrow |S_c| \cap |S_{c'}| = \phi$ for $c \neq c'$.

(3) $\Rightarrow p \in S_{c^*}$, $c^* = \min \left\{ \begin{array}{l} c \in \Gamma \text{ s.t.} \\ p \in |\mathcal{X}_{\leq c}| \end{array} \right\}$

(2) + (3) $\Rightarrow \exists!$ of "HN. filtration". $\boxtimes$

Let $\mathcal{X}$ verify (+) w/ weak Θ-stratif-

$\mathcal{X}_{\leq c} \xhookrightarrow{\circ} \mathcal{X} \rightsquigarrow \mathrm{Filt}(\mathcal{X}) \xhookrightarrow{\circ} \mathrm{Filt}(\mathcal{X})$

$\rightsquigarrow S_c \xhookrightarrow{\circ} \mathrm{Filt}(\mathcal{X}_{\leq c}) \xhookrightarrow{\circ} \mathrm{Filt}(\mathcal{X})$

$\rightsquigarrow$ (2) + (3) $\Rightarrow |\mathcal{X}_{\leq c}| = |\mathcal{X}| \setminus \bigcup_{c' > c} ev_1(S_{c'})$

$\rightsquigarrow$ Θ-stratif. is encoded in

$$S := \bigsqcup_{c > \{-\infty\}} S_c \xhookrightarrow{\theta} \mathrm{Filt}(\mathcal{X})$$

Θ-stratif data $\iff$ $\begin{cases} (1') & S \xhookrightarrow{\theta} \mathrm{Filt}(\mathcal{X}) \\ (2') & \mu : S \longrightarrow \Gamma \\ & \quad S_c \mapsto c \end{cases}$

$$S_c \xhookrightarrow{\theta} \mathrm{Filt}(\mathcal{X}_{\leq c}) \xhookrightarrow{\theta} \mathrm{Filt}(\mathcal{X})$$

$\rightsquigarrow$

$$\mathrm{Irred}(S_c) \subset \mathrm{Irred}(\mathrm{Fil}(\mathcal{X}_{\leq c})) \subset \mathrm{Irred}(\mathrm{Filt}(\mathcal{X}))$$

Θ-stratif data $\overset{\star\star}{\iff}$ $\begin{cases} (1'') & \mathrm{Irred}(S) \subset \mathrm{Irred}(\mathrm{Filt}(\mathcal{X})) \\ (2'') & \mu : S \longrightarrow \Gamma \\ & \quad S_c \mapsto c \end{cases}$

$\overset{\text{extend}}{\rightsquigarrow}$ $(2'')$ to a map

$$\mu : |\mathrm{Filt}(\mathcal{X})| \longrightarrow \Gamma \sqcup \{-\infty\}$$

$\mu(b) = \max\left(\{-\infty\} \sqcup \{\mu(s),\ b \text{ lies in an irred. component}\right.$

$s \in \mathrm{Irred}(S)$)

__Def:__ __stability function__

$$M^\mu(p) = \sup\{\mu(f) : f \in |\mathrm{Filt}(\mathcal{X})|,\ f(1) = p\} \in \Gamma \sqcup \{-\infty\}$$

$\rightsquigarrow$ p is __unstable__ if $M^\mu(p) > -\infty$, __semistable__ o/w

__Th:__ Let $\mathcal{X}$ an alg. stack verifying $(\dagger)$ with data in $\star, \star$

Then the data in $\star\star$ define a weak Θ-stratif iff

① __HN-property__: $\forall p \in \mathcal{X}(k)$ unstable of finite type, $\exists !\, f \in |\mathrm{Flag}(p)|$ lying over an irreducible component in S with $\mu(f) = M^\mu(p)$.
(f is the HN-filtration of p)

② __HN-specialization__: $\forall$ VR R, $K = \mathrm{Frac}(R)$, $k = R/\mathfrak{m}$, $\forall\, \xi : \mathrm{Spec}\, R \to \mathcal{X}$ whose generic point is unstable and a HN-filtration of $\xi(K)$ $f_K \in \mathrm{Flag}(\xi)(K)$, we have $\mu(f_K) \leq M^\mu(\xi|_{\mathrm{Spec}(k)})$

③ __open strata__: If $\mathrm{Spec}\, R \longrightarrow \mathrm{Filt}(\mathcal{X})$ is a map from a DVR essentially of finite type over B whose special point is a HN-filtration, then its generic point is also a HN-filtration as well.

④ __Local finiteness__: $\forall$ T a scheme of finite type, $\forall\, \varphi : T \longrightarrow \mathcal{X}$, $\exists$ finite subset of S such that every unstable finite type point in T has a HN-filtration lying on one of these irreducible components.

⑤ __semi-continuity__: If $f : \Theta_k \longrightarrow \mathcal{X}$ is a HN-filtration for $f(1)$, then $M^\mu(f(0)) \leqslant \mu(f)$.

Def: • T is a B-stack, the stack of $\mathbb{Z}^m$-filtered objects,

$\mathrm{Filt}^m(\mathcal{X}) := \underline{\mathrm{Map}}(\Theta^m, \mathcal{X})$

• $f \in \mathrm{Filt}^m(\mathcal{X})(k)$ is non-degenerate if

$(\mathbb{G}_m)^m_k \longrightarrow \mathrm{Aut}(f(0))$ of group sheaves over

$\mathrm{Spec}\, k$ has finite kernel.

• The component Fam is $CF(\mathcal{X})_\bullet$

$CF(\mathcal{X})_m := \{ \text{non-deg. } \alpha \in \pi_0(\mathrm{Filt}^m(\mathcal{X})) \}$

• The component space: $\mathrm{Comp}(\mathcal{X}) := \mathbb{P}(CF(\mathcal{X})_\bullet)$

• A numerical invariant on $\mathcal{X}$ is a continuous

function $\mu : \mathcal{U} \subset \mathrm{Comp}(\mathcal{X}) \longrightarrow \mathbb{R}$

• Stability function: $M^\mu : |\mathcal{X}| \longrightarrow \mathbb{R} \sqcup \{-\infty\}$

$p \mapsto \sup\{ \mu(f) \mid f \in \mathcal{U}, f(1) = p \in |\mathcal{X}| \}$

• Degeneration space: $\mathrm{Deg}(\mathcal{X}, p) := \mathbb{P}(DF(\mathcal{X}, p)_\bullet)$

• $\mu : \mathcal{U} \longrightarrow \mathbb{R}$ is (locally-strictly) q-concave

if $\forall k$, $\forall p \in \mathcal{X}(k)$

$$\mathcal{U}_p^{\mu>0} := \{ x \in \mathcal{U} \mid \mu(x) > 0 \} \subset \mathrm{Deg}(\mathcal{X}, p)$$

is (locally) convex and $\mu|_{\mathcal{U}_p^{\mu>0}}$ is (strictly) q-concave.

- $f_1, f_2 \in \mathrm{Filt}(\mathcal{X})(k)$ are antipodal if $\exists$

$$\begin{tikzcd}
\Theta_k \arrow[dr, "g_1"] \arrow[rr, bend left, "f_1"] & & \\
 & (pt/\mathbb{G}_m)_k & \mathcal{X} \\
\Theta_k \arrow[ur, "g_2"] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

s.t. the cocharacters $(\mathbb{G}_m)_k \longrightarrow (\mathbb{G}_m)_k$ induced by g_1, g_2 have opposite signs.

- $\mu : \mathcal{U} \subset \mathrm{Comp}_p(\mathcal{X}) \longrightarrow \mathbb{R}$ is a standard numerical inv. if μ is loc. str. q-concave, and $\mathcal{U}^{\mu>0}$ doesn't contain a pair of antipodal points.

<u>HN problem</u> : Given $p \in \mathcal{X}(k)^{us}$, $\exists !?$ rational point

$f \in \mathcal{U}_p \subset \mathrm{Deg}(\mathcal{X}, p)$ maximizing $\mu(f)$?

<u>Def:</u> let $b \in H^4(\mathcal{X}, \mathbb{R})$ be positive definite.

let $l \in H^2(\mathcal{X}, \mathbb{R})$.

The numerical inv. associated to (l, b) is the pair

$$\begin{cases} \mathcal{U} := \{ x \in \mathrm{Comp}(\mathcal{X}) \mid \tilde{b}(\tilde{x}) > 0 \} \\ \mu(x) := \dfrac{\tilde{l}(\tilde{x})}{\sqrt{\tilde{b}(\tilde{x})}} \end{cases}$$

where $\tilde{x} \in |CF(\mathcal{X})_0|$ is a lift of $x \in \mathrm{Comp}(\mathcal{X})$

<u>E.g.:</u> Reductive $G \curvearrowright X$ proj. curve over k.

Let $\mathcal{L}$ be a G-linear ample line bdle

$$\left(f : [\mathbb{A}^1/\mathbb{G}_m] \longrightarrow \mathrm{Bun}_G(X) \right) \overset{\text{Rees}}{\longleftrightarrow} \left(\begin{array}{l} \text{1-PS } \lambda : \mathbb{G}_m \to G \\ \text{s.t. } \lim_{t \to 0} \lambda(t) \text{ exists} \\ \text{up to conjugation by} \\ \text{an elt of } P_\lambda(k) \end{array} \right)$$

Choose $l \in H^2(\mathcal{X}, \mathbb{Q})$, $b \in H^4(\mathcal{X}, \mathbb{Q})$ positive definite

s.t. $\mu(f) =$

Def of Kempf-Ness stratification in GIT : Reductive $G \curvearrowright X$ proj. var. over k

$\mathcal{L}$: G-linearized ample line bundle on X

For $x \in X$ a point associate to $\forall$ nontrivial 1-PS $\lambda : \mathbb{G}_m \to G$ s.t. $\lim_{t \to 0} \lambda(t).x$ exists

a normalized numerical invariant $\mu^L(x, \lambda)$.

x unstable $\Longrightarrow$ $\exists$ stratum $S_\lambda \ni x$

s.t. λ maximizes $\mu^{\mathcal{L}}(x,\lambda)$ and the connected component of

$X^\lambda = \{ x \in X \mid \lambda(\mathbb{G}_m) \subset G_x \}$ in which $\lim_{t\to 0} \lambda(t).x$ lies.

In Θ-instability theory

(Th. 1.21) $(x,\lambda) \longleftrightarrow f : \Theta \to \mathcal{X}$ so that S_λ is determined

by f which maximizes μ (analogue of HM num. inv.) if $\exists$ an iso $f(1) \cong x$

$\mu = \mu(\ell, b)$ s.t. $\ell \in H^2(\mathcal{X})$, $b \in H^4(\mathcal{X})$

and roughly : $\mu : \pi_0[\Theta, \mathcal{X}] \to \mathbb{R}$

$\ell \in H^2(\mathcal{X}, \mathbb{R})$: plays role of $c_1(\mathcal{L})$ where $\mathcal{L}$ is the G-linearized ample line bdl in GIT.

$b \in H^4(\mathcal{X}, \mathbb{R})$: ——— of an invariant positive definite inner product on the Lie algebra of the compact form of G.

Def: The numerical invariant associated to (ℓ, b) is the pair

$$U := \{ x \in \mathrm{Comp}(\mathcal{X}) \mid \hat{b}(\tilde{x}) > 0 \}, \quad \mu(x) := \frac{\hat{\ell}(\tilde{x})}{\sqrt{\hat{b}(x)}}$$

where $\tilde{x} \in |CF(\mathcal{X})_\bullet|$ is some lift of $x \in \mathcal{C}omp(\mathcal{X})$.

- $b \in H^4(\mathcal{X}, \mathbb{R})$ is <u>positive definite</u> if

  $\forall\ \gamma : pt/\mathbb{G}_m \longrightarrow \mathcal{X}$ non degenerate,

  $\gamma^*(b) \in H^4(pt/\mathbb{G}_m) \simeq \mathbb{A}.u^2 \subset \mathbb{R}[u]$ is a positive multiple of u^2.

$\rightsquigarrow$ Define the unstable strata $S_c \subset [\Theta, \mathcal{X}]$ to consist of points $p \in |\mathcal{X}|$ for which $\mu > 0$ and μ is maximal for a point in the fiber $ev_1^{-1}(p)$.

(unique up to $\mathbb{N}^*$-action).

<u>Rem</u>: • $\mathbb{N}^* \times \pi_0[\Theta, \mathcal{X}] \longrightarrow \pi_0[\Theta, \mathcal{X}]$ | d_m: n-fold ramified covering
$(m, f) \longmapsto f \circ d_m$ | map $\Theta \longrightarrow \Theta$

⚠ • Any $(\ell, b) \in H^2(\mathcal{X}) \times H^4(\mathcal{X}) \rightsquigarrow \mu$, but the strata S_c defined by μ doesn't always give rise to a Θ-stratif.

- in GIT, fix a positive definite coh class $b \in H^4(BG)$

  $\rightsquigarrow$ recover instability w.r.t. an invertible sheaf L s.t. $c_1(L) = \ell$

- <u>Degeneration space</u>:

$ev_1^{-1}(p)$ is an infinite set of connected components so finding a maximum of μ is intractable problem $\rightsquigarrow$ the Degeneration space $\mathcal{D}(\mathcal{X}, p)$

<u>idea</u>:

$\forall p \in \mathcal{X}(k)$, a large top. space $\mathcal{D}(\mathcal{X}, p)$: degeneration space of p, s.t.

$ev_1^{-1}(p)(k) \longleftrightarrow$ dense set of "rational points" in $\mathcal{D}(\mathcal{X}, p)$

so $(\ell, b) \in H^2(\mathcal{X}) \times H^4(\mathcal{X}) \rightsquigarrow \mu \overset{\text{extends to}}{\rightsquigarrow}$ continuous fct on $\mathcal{U} \overset{\text{open}}{\subset} \mathcal{D}(\mathcal{X}, p)$.

E.g.: $\mathcal{X} = \mathbb{A}^2/\mathbb{G}_m^2$, $p = (1,1)$,

$\forall (a,b) \in \mathbb{Z}^2_{\geq 0} \rightsquigarrow$ defines $\mathbb{G}_m \xrightarrow{\varphi} \mathbb{G}_m^2$ $\overset{\text{extends to}}{\rightsquigarrow}$ map of quotient stacks $f : \Theta \longrightarrow \mathbb{A}^2/\mathbb{G}_m^2$ with an iso $f(1) \simeq (1,1)$

$(ma, mb) \rightsquigarrow f' = f \circ d_m$ | $d_m : \Theta \longrightarrow \Theta$ the m-fold ramified cover

$$\left(\begin{array}{c} \text{Rational rays in} \\ \text{the cone } (\mathbb{R}_{\geq 0})^2 \end{array} \right) \longleftrightarrow \left(\begin{array}{c} [f] \text{ (up to ramified covering)} \\ \text{of } f : \Theta \to \mathbb{A}^2/\mathbb{G}_m \text{ s.t. } f(1) \simeq (1,1) \end{array} \right)$$

$$\updownarrow$$

$\mathbb{Q} \cap \mathcal{D}(\mathcal{X}, p) \mid \mathcal{D}(\mathcal{X}, p) = [0,1]$

So "rational points" in $\mathcal{D}(\mathcal{X}, p)$ parametrise a "family" of maps $\Theta \longrightarrow \mathbb{A}^2/\mathbb{G}_m^2$ up to ramified coverings.

HN Problem: Given $p \in \mathcal{X}(k)^{us}$, $\exists 1?$ rational point $f \in \mathcal{U} \subset \mathcal{D}eg(\mathcal{X}, p)$ maximizing $\mu(f)$?

E.g.: let a reductive $G \curvearrowright X$ projective curve over k, let ℓ be G-linear. ample line bdle on X

$$\left(f : [\mathbb{A}_1 / \mathbb{G}_m] \longrightarrow \mathrm{Bun}_G(X) \right) \underset{\text{construction}}{\overset{\text{Recs}}{\Longleftrightarrow}} \left(\begin{array}{l} \bullet \text{ 1-PS } \lambda : \mathbb{G}_m \to G \\ \text{up to conjugation by} \\ \text{an elem. of } P_\lambda(k) \\ \bullet \text{ principle } P_\lambda\text{-bundle over } X \end{array} \right)$$

can choose $\ell \in H^2(\mathcal{X}, \mathbb{Q})$, and $b \in H^4(\mathcal{X}, \mathbb{Q})$ positive definite s.t.

$$\mu(f) = \frac{f^* \ell}{\sqrt{f^* b}} = \mu(P, \lambda) \text{ is the normalized HM numerical invariant.}$$

(R): For any rational simplex $\Delta \longrightarrow \mathcal{U} \subset \mathrm{Comp}(\mathcal{X})$ if $\mu|_\Delta$ has a point with $\mu > 0$, then μ has a maximum at a <u>rational point</u> of Δ.

<u>Def</u>: Δ_{n-1} is the standard $(n-1)$ simplex realized as the space of Rays in $(\mathbb{R}_{\geq 0})^n$, so $\Delta_{n-1} := \left((\mathbb{R}_{\geq 0})^n - \{0\} \right) / (\mathbb{R}_{>0})^\times$

we call a <u>rational simplex</u> a map $\Delta_{n-1} \longrightarrow \mathrm{Comp}(\mathcal{X})$.

if $n = 1$ we call a <u>rational point</u> a map $\Delta_0 \longrightarrow \mathrm{Comp}(\mathcal{X})$

<u>Lemma</u>: Any numerical invariant associated to $(\ell, b) \in H^2(\mathcal{X}, \mathbb{Q}) \times H^4(\mathcal{X}, \mathbb{Q})$ satisfies (R) $\boxtimes$

Now we give necessary and sufficient conditions for a standard numerical invariant to define a Θ-stratification:

<u>Th</u>: let $\mathcal{X}$ verify $(\dagger)$, let $\mu : \mathcal{U} \subset \mathrm{Comp}(\mathcal{X}) \longrightarrow \mathbb{R}$ be a standard numerical invariant satisfying (R) for which

skip

$\mathcal{U}C\,\mathcal{C}omp(\mathcal{X})$ is closed. Then μ defines a weak Θ-stratification iff it satisfies:

i) uniqueness part of the HN-property

ii) HN-specialization property

iii) condition (B_2)

These conditions can be greatly simplified when we consider the notion of Θ-reductive stack:

Def: $\mathcal{X}$ is Θ-reductive if it verifies the valuative criteria for properness, which means:

$\forall$ R valuation ring with quotient field K,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Filt}(\mathcal{X}) \arrow[d, "ev_1"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathcal{X} \end{tikzcd}$$

commutes. This is the usual valuative criterion for maps of schemes. It means that for any family over $\operatorname{Spec} R$, any filtration of the generic point extends uniquely to a filtration of the family.

Th: Let $\mathcal{X}$ Θ-reductive verifying $(\dagger)$, let $\mu : \mathcal{U}C\,\mathcal{C}omp(\mathcal{X}) \to \mathbb{R}$ be a standard numerical invariant satisfying (R) for which $\mathcal{U}C\,\mathcal{C}omp(\mathcal{X})$ is closed. Then μ defines a weak Θ-stratification iff it satisfies (B_2)

(B_2): for any map from fin. type aff. scheme

$\xi : T \to \mathfrak{X}$, $\exists$ a q-compact substack $\mathfrak{X}' \subset \mathfrak{X}$ s.t. $\forall$ finite type points $p \in T(k)$, $\forall$ $f \in Flag(p)$ with $\mu(f) > 0$, $\exists$ $f' \in Flag(p)$ with $\mu(f') \geq \mu(f)$ and $gr(f') \in \mathfrak{X}'$

Recall:

- Flag space: Given a map $T \xrightarrow{\xi_T} \mathfrak{X}$ over B,

$$Flag(\xi_T) := \underline{Map}(\Theta, \mathfrak{X}) \times_{ev_1, \mathfrak{X}, T} T$$

STOP Here! thank you!

Let X a projective k-scheme

<u>Def:</u> • A <u>stability condition</u> on X is a pair $(\mathcal{A}, Z)$ where
$\mathcal{A}$ is abelian cat, $Z : K(\mathcal{A}) \longrightarrow \mathbb{H} \cup \mathbb{R}_{\leq 0}$

• For $E \in ob(\mathcal{A})$, <u>the phase</u> $\phi(E) \in]0,1]$
is the unique number s.t. $Z(E) = |Z(E)| e^{2i\pi\phi(E)}$,
$\phi = 1$ if $Z(E) = 0$ by convention

Let $E^{\phi \geq \varepsilon} \subset E$ denote the largest subobject in a HN-filtrat. with phase $\geq \varepsilon$.

• $E \in ob(\mathcal{A})$ is <u>slope semistable</u> if
$\nexists F \subsetneq E$ in $\mathcal{A}$ s.t. $\phi(F) > \phi(E)$.

• $(Z, \mathcal{A})$ <u>has the HN property</u> if $\forall E \in ob(\mathcal{A})$,
$\exists\ E = E_1 \supset \dots \supset E_p \supset E_{p+1} = 0$ in $\mathcal{A}$ s.t.
$gr_i E_\bullet$ is semistable and $\phi(gr_i E_\bullet)$ increasing in i.

• $E \in ob(\mathcal{A})$ is <u>torsion</u> if $Z(E) \in \mathbb{R}_{\leq 0}$
———— is torsion-free if $\nexists F \subsetneq E$: F torsion

Denote $\mathcal{T} \subset \mathcal{A}$ the full subcat of torsion objects
—— $\mathcal{F} \subset \mathcal{A}$ ———— torsion-free ——

• $\forall (z_1, \dots, z_p)$ a sequence with $z_i \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$
the convex polyhedron $Pol(\{z_j\})$ = convex hull of
the points $z_p, z_p + z_{p-1}, \dots, z_p + \dots + z_1$.

$\forall E \in ob(\mathcal{A})$, $Pol^{HN}(E) := Pol(E_\bullet) = pol(\{Z(gr_i E_\bullet)\})$

Th: Let $(\mathcal{A}, Z)$ be a <u>stability condition</u> on X

Let $\mu = (\ell, b)$ be associated to the cohomology classes

$$\ell := |Z(v)|^2 c_1\left((p_1)_*\left(\mathcal{E} \otimes p_2^* \mathcal{I}\left(-\frac{w_Z}{Z(v)}\right)\right)\right) \in H^2(\mathcal{M}, \mathbb{R})$$

$$b := 2c_2\left((p_1)_*\left(\mathcal{E} \otimes p_2^* \mathcal{I}(w_Z)\right)\right) \in H^4(\mathcal{M}, \mathbb{R})$$

where $\mathcal{E}$ is the universal object in the derived category of $\mathcal{M} \times X$

and p_1, p_2 are the projections $\mathcal{M} \xleftarrow{p_1} \mathcal{M} \times X \xrightarrow{p_2} X$,

and $w_Z \in K^0(\mathrm{Perf}(X)) \otimes \mathbb{C}$ where $\mathrm{perf}(X)$ is formed by perfect complexes,

is a class it's the triangulated subcategory of the derived category of coherent sheaves on X.

0) Then $E \in \mathrm{ob}(\mathcal{F})$ is slope semistable iff $M^\mu([E]) \leq 0$.

Moreover, TFAE:

1) $(\mathcal{A}, Z)$ has the HN property

2) every object in $\mathcal{A}$ has a maximal torsion subobject, and for every unstable $E \in \mathrm{ob}(\mathcal{F})$, $\mu : \mathcal{U}_E \subset \mathrm{Deg}(\mathcal{M}, E) \to \mathbb{R}$ obtains a maximum.

If $\mathcal{A}$ is Noetherian, (1) and (2) are equivalent to

3) $\forall E \in \mathrm{ob}(\mathcal{A})$, $\{\phi(F) \mid F \subset E\}$ has a maximal element.

Proof sketch :

0) Let $F \subsetneq E$ s.t. $\phi(F) > \phi(E)$, and take the 2-step filt.

$gr_2 E_\bullet = F$, $gr_1 E_\bullet = E/F$, $w_2 > w_1$ arbitrary

E torsion-free $\Rightarrow Z(F) \neq 0$

So by straightforward computations we can see that

$$\frac{1}{q} b^* l = w_1 \mathcal{I}\left((Z(E) - Z(F))\overline{Z(E)}\right) + w_2 \mathcal{I}\left(Z(F)\overline{Z(E)}\right)$$

$$= (w_2 - w_1)\, \mathcal{I}\left(Z(F)\overline{Z(E)}\right) > 0$$

$$\Rightarrow M^\mu(E) > 0.$$

before proving the converse, we need a useful

Claim: $\forall$ descending weighted filtration $(E_\bullet, w_\bullet)$ $\exists$ a sequence of deletions resulting in a descending weighted filtration $(E'_\bullet, w'_\bullet)$ which is convex in the sense that $(\phi'_1 < \dots < \phi'_{p'})$, such that $Pol(E_\bullet) = Pol(E'_\bullet)$ and $\mu(E'_\bullet) \geq \mu(E_\bullet)$ with strict inequality if $E_\bullet$ is not convex.

Proof: follows from computations on insertion / deletion in the rank-degree-weight sequence of a filtration $\square$

Conversely, if $f : \Theta \to \mathcal{M}$ corresponds to a weighted filtration $E_\bullet$ s.t. $\mu(f) > 0$.

By claim, , $\exists E'_\bullet$ a convex filtration s.t.

$\mu(E'_\bullet) \geq \mu(E_\bullet) > 0$ $\qquad \phi'_1 < \phi'_2 < \dots < \phi'_{p'}$

$\rightsquigarrow$ $E'_\bullet$ is non trivial

$\rightsquigarrow$ the first object $E'_{p'} \subset E$ destabilizes E.

1) $\Rightarrow$ 2) : If $(\mathcal{A}, Z)$ has HN property then $E^{\phi \geq 1} \subset E$ maximal torsion subsheaf.

(Note that torsion objects are automatically semistable so we only need to check torsion-free objects. why

Take $E \in ob(\mathcal{F})$ unstable.

claim $\rightsquigarrow$ $\exists$ a set of convex filtrations.

$\rightsquigarrow$ It suffices then to maximize μ over the set of convex filtrations, which follows easily from linear algebra and provides an integral formula for this maximum.

2) $\Rightarrow$ 1) Sps any $E \in ob(\mathcal{A})$ has a maximal torsion subobject $T \subset E$, then T is semistable.

suffices $\rightsquigarrow$ E/T (which is torsion-free) admits HN-filtration with phases < 1.

Assume $E \in Ob(\mathcal{F})$ unstable.

choose a point $pt \in Deg(\mathcal{M}, E)$ maximizing μ

s.t. $pt \longleftrightarrow$ descending filtration $E_1 \supset \dots \supset E_m$.

+ real weights

$\omega_1 \leq \dots \leq \omega_m$

(Actually the degeneration space can be identified with the space of finite real weighted filtration of E up to positive rescaling of weights) (see Rem. 5.16 p 134 HLPC)

claim $\Longrightarrow$ $\phi_1 < \dots < \phi_p \boxed{< 1}$, $\phi_i := \phi(E_i / E_{i+1})$

Fix $F \subset E_i / E_{i+1}$ and refine $E_\bullet$ to obtain

$$E'_\bullet = (E_m \subset \dots \subset E_{i+1} \subset \hat{F} \subset E_i \subset \dots \subset E_1)$$

preimage of F under $E_i \to E_i / E_{i+1}$

$\rightsquigarrow$ $Pol(E_\bullet) \subsetneq Pol(E'_\bullet)$ if $Z(F) \neq 0$ and $\phi(F) > \phi(E_i / E_{i+1}) = \phi_i$

But $\mu(E_\bullet)$ is maximal so claim $\Longrightarrow$ $Pol(E_\bullet) = Pol(E'_\bullet)$

So $Z(F) = 0$ or $\phi(F) \leq \phi_i$

So $gr_i(E)$ is either torsion-free $\rightsquigarrow$ semistable,

or the maximal torsion subobj $F \subset gr_i(E_\bullet)$ has $Z(F) = 0$

inductive $\rightsquigarrow$ procedure
- redefine E_{i+1} = preimage of F
- construct new filtration of same length

$E'_1 \supset \dots \supset E'_m$ with $Z(E'_i) = Z(E_i)$ $\forall i$, and

E_i / E_{i+1} torsion-free.

$\rightsquigarrow$ • $E'_\bullet$ maximizes μ $\rightsquigarrow$ E_i / E_{i+1} is semistable in increasing phase

$\Rightarrow E'_\bullet$ is a HN filtration for E.

when $\mathcal{A}$ is Noetherian similar arguments give

(1) $\Leftrightarrow$ (3) is classical. can be found in other contexts $\boxed{}$